

Indian Statistical Institute
Functional Analysis. Final Exam
M.Math First Year. 2025-2026

Time : 3 Hours Date : 24.04.2026 Maximum Marks : 100 Instructor : Jaydeb Sarkar

(i) Answer all questions. (ii) m = the Lebesgue measure on \mathbb{R} .

- (1) (15 marks) Let A be a bounded linear operator on a Banach space X . Suppose that $A^7 \neq 0$ and $A^{11} = 0$. Compute $\sigma(A^9)$.
- (2) (15 marks) Define the shift operator S on $l^2(\mathbb{Z}_+)$ by $Se_n = e_{n+1}$ for all $n \in \mathbb{Z}_+$. Compute $\sigma(S)$. Also, prove that the point spectrum of S is empty.
- (3) (15 marks) Let X be a Banach space. Prove that the set of compact operators on X forms a two-sided ideal in the algebra of bounded linear operators on X . Prove that S on $l^2(\mathbb{Z}_+)$ is not compact.
- (4) (15 marks) Let \mathcal{H} be a Hilbert space, $\{x_n\} \subseteq \mathcal{H}$, and let $x \in \mathcal{H}$. Assume that x_n converges weakly to x . Prove that $x_n \rightarrow x$ in \mathcal{H} if and only if $\|x_n\| \rightarrow \|x\|$.
- (5) (15 marks) Let $\{e_1, \dots, e_n\}$ be a linearly independent subset of a Banach space X . Show that there are $\{\varphi_1, \dots, \varphi_n\} \subseteq X^*$ such that

$$\varphi_i(e_j) = \delta_{ij},$$

for all $i, j = 1, \dots, n$.

- (6) (15 marks) Let $p = 1$ or $p = \infty$, and consider the Banach space $C[0, 1]$ equipped with the sup norm. Prove that $C[0, 1]$ is a linear subspace of $L^p([0, 1], m)$. Are the sup norm and L^p norm on $C[0, 1]$ equivalent?
- (7) (10 + 5 + 5 = 20 marks) Consider $L^2[0, 1]$ with respect to m . Define the Volterra operator $V : L^2[0, 1] \rightarrow L^2[0, 1]$ by

$$(Vf)(x) = \int_0^x f(t) dm(t).$$

Compute the adjoint V^* and prove that $V + V^*$ is a rank one operator. Also prove that $V + V^*$ is an orthogonal projection.